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Space-time dependent couplings in $\mathcal{N} = 1$ SUSY gauge theories: anomalies and central functions

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Abstract: We consider $\mathcal{N} = 1$ supersymmetric gauge theories in which the couplings are allowed to be space-time dependent functions. Both the gauge and the superpotential couplings become chiral superfields. As has recently been shown, a new topological anomaly appears in models with space-time dependent gauge coupling. Here we show how this anomaly may be used to derive the NSVZ β-function in a particular, well-determined renormalisation scheme, both without and with chiral matter. Moreover we extend the topological anomaly analysis to theories coupled to a classical curved superspace background, and use it to derive an all-order expression for the central charge c, the coefficient of the Weyl tensor squared contribution to the conformal anomaly. We also comment on the implications of our results for the central charge a expected to be of relevance for a four-dimensional C-theorem.

Keywords: Renormalization Group, Anomalies in Field and String Theories, Superspaces.

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Contents

1. Introduction

Space-time dependent or 'local' couplings $\lambda^i = \lambda^i(x)$ are a standard tool in quantum field theory. They may be viewed as sources for composite operators. Well-defined operator insertions are obtained by functionally varying the generating functional with respect to the local couplings,

$$\frac{\delta}{\delta\lambda^i(x)} W = \langle \mathcal{O}_i(x) \rangle \,. \tag{1.1}$$

The concept of local couplings is particularly appealing in supersymmetric theories where holomorphy is at the origin of non-renormalisation theorems [1 – 4]. As we will see below, new results for supersymmetric gauge theories are obtained by promoting the couplings, both the gauge couplings as well as the superpotential couplings, to full chiral and antichiral space-time dependent superfields $\lambda(z)$ and $\bar\lambda(\bar z)$, respectively, satisfying $\bar D_{\dot\alpha} \lambda = 0$ and $D_\alpha \bar\lambda = 0$.

The consequences of allowing for local supercouplings in $\mathcal{N} = 1$ theories have recently been investigated in a series of papers [5]–[10] within a perturbative approach. Both a component [6] and a superspace [7] approach were taken to study pure $\mathcal{N} = 1$ super Yang-Mills theory. The Wess-Zumino model was considered in [8].

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Most importantly, for pure $\mathcal{N}=1$ gauge theory it was shown in these publications that in the presence of local couplings there is an additional new anomaly. This anomaly appears in the Ward identity for the topological symmetry associated with the theta angle and manifests itself in an anomalous divergence of the topological current. At one loop, this anomaly is given by [7]

$$\left(\int d^6z\,\frac{\delta}{\delta\lambda}-\int d^6\bar{z}\,\frac{\delta}{\delta\bar{\lambda}}\right)\Gamma=\frac{\mathcal{A}_1}{4}\left(\int d^6z\,\frac{1}{\lambda'}\,\mathrm{tr}(W^\alpha W_\alpha)-\int d^6\bar{z}\,\frac{1}{\bar{\lambda}'}\,\mathrm{tr}(\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha})\right), \quad (1.2)$$

where $\lambda'(z) = \lambda(z) + 1/2g^2$ and $\bar{\lambda}'(\bar{z}) = \bar{\lambda}(\bar{z}) + 1/2g^2$. Γ is the vertex functional of pure $\mathcal{N}=1$ Yang-Mills theory.[1] Moreover the one-loop coefficient $\mathcal{A}_1$ is given by $\mathcal{A}_1 = C_2(G)/8\pi^2$. The l.h.s. of (1.2) is the symmetry transformation of the vertex functional under the topological symmetry. The r.h.s. is the new anomaly which vanishes in the limit of constant couplings where the integrand becomes the Pontryagin density.

The one-loop coefficient of the anomaly in (1.2) has been calculated in [6] in a component approach. In fact, an alternative way of obtaining an equivalent expression for the lowest component of (1.2) is to vary the component action with respect to the space-time dependent theta angle. This gives rise to the one-loop result [6, 12]

$$\frac{\delta}{\delta\tilde{\theta}(x)}\,\Gamma\;=\;-\,(1\,+\,\mathcal{A}_1\;g^2)\left(\frac{1}{4}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}+\partial^\mu(\bar{\lambda}^a\sigma_\mu\lambda^a)\right)(x)\,,\qquad \tilde{\theta}=\frac{\theta}{8\pi^2}\,. \quad (1.3)$$

Here ${\lambda_\alpha}^a$ are the gauginos. (1.3) implies that the divergence of the topological current J_μ, defined by

$$J_\mu=\tfrac{1}{8}\,\varepsilon_{\mu\sigma\nu\rho}(A^{\sigma a}\partial^\nu A^{\rho a}+\frac{1}{3}A^{\sigma a}A^{\nu b}A^{\rho c}f_{abc})+\bar{\lambda}^a\sigma_\mu\lambda^a\,, \quad (1.4)$$

is anomalous. In fact, classically we have $\partial^\mu J_\mu = \frac{1}{4}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \partial^\mu(\bar{\lambda}^a\sigma_\mu\lambda^a)$ and therefore (1.3) may equivalently be written as

$$\frac{\delta}{\delta\tilde{\theta}}\,\Gamma=-\,(1+\mathcal{A}_1\,g^2)\,\partial^\mu J_\mu\,,\qquad \mathcal{A}_1=\frac{C_2(G)}{8\pi^2}\,, \quad (1.5)$$

where the anomalous contribution is $\mathcal{A}_1 g^2\,\partial^\mu J_\mu$.

In [7], the anomaly in (1.2) is used to obtain a general scheme independent result for the gauge β function to all orders. The main point in this derivation is to add an appropriate counterterm to the action which shifts the anomaly from the topological Ward identity (1.2) to the Callan-Symanzik equation. There is an interesting analogy between this anomaly shift and the approach of [13], where anomalous transformations of the path integral functional measure are considered. - Note in particular that the one-loop coefficient $\mathcal{A}_1 = C_2(G)/8\pi^2$ is the coefficient which appears in the denominator of the NSVZ β-function [14].

[1] Γ is the vertex functional of the BPHZ approach. Within perturbation theory, by virtue of the so-called 'action principle', Γ is equivalent to the 'quantum effective action' Γ_{eff}. This in turn is a local function of the fields and of the couplings, constructed order by order in perturbation theory. In this sense Γ_{eff} corresponds to the bare action in the standard perturbative approach. - A review of the BPHZ approach may be found in [11].

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A further use of local couplings, so far used for non-supersymmetric field theories or for SUSY theories in components, is that they allow for an elegant formulation of *local* renormalisation group (RG) or Callan-Symanzik (CS) equations. These determine how conformal symmetry is broken in a quantised field theory. This is in contrast to the standard (or *global*) RG equations, which express the breakdown of scale invariance upon quantisation. For the formulation of local RG equations it is necessary to couple the quantum field theory to a classical curved space background, with the metric acting as source for the energy-momentum tensor [15, 16] (see also [17]). The significance of this approach is that it may be of relevance for a proof of an analogue of the Zamolodchikov C-theorem [18] in more than two dimensions. In particular in [15] an alternative derivation of the two-dimensional C-theorem was given using local couplings. It is essential for this analysis that in the presence of local couplings, there are new conformal anomalies in addition to the familiar anomalies involving the curvature of the background metric. These new conformal anomalies involve derivatives of the couplings. Furthermore in [15] within the local coupling approach, the flow of a candidate a-function (related to the coefficient of the Euler anomaly) for a possible four-dimensional C-theorem [19] is related to a quadratic form, which however so far has not been shown to be positive definite.

A related approach to supersymmetric theories was used in [20] and in particular in [21] where a four-loop expression for the candidate a-function was given and shown to coincide with non-perturbative results for $a_{\rm UV} - a_{\rm IR}$ found in [22]. These non-perturbative results were obtained using 't Hooft anomaly matching and by performing explicit one-loop computations of three-point correlators involving the R current, the energy-momentum tensor and a particular anomaly-free current.

A different approach to a possible four-dimensional C-theorem for supersymmetric theories is the principle called 'a maximization' which has recently been proposed and investigated in [23]–[28]. The local coupling approach taken here is complementary to a maximization, though relations between the two approaches exist [25].

The purpose of this paper is twofold. First we consider $\mathcal{N} = 1$ SUSY gauge theories with local chiral supercouplings both without and with chiral matter. We show how the NSVZ β-function may easily be derived from the topological anomaly present for local couplings. This β-function is naturally associated to a particular renormalisation scheme which we describe in detail.

Secondly we consider $\mathcal{N} = 1$ SUSY gauge theories with local couplings which in addition are coupled to a classical supergravity background in superspace. The aim of this analysis is to find new results for the coefficients of the gravitational anomalies. - For the superspace formulation of local CS equations, the supercurrent associated to superconformal transformations has to be coupled to the appropriate superspace supergravity field [29, 30]. An analysis of local CS equations for supersymmetric theories with constant coupling has been given in [31 − 33]. Here we extend this approach to the local coupling case. The resulting superconformal Ward identities may be viewed as a generalisation of previous results for the anomalous divergence of the supercurrent [34 − 37] to the off-shell case with curved superspace background. With this approach we are able to give an all-order derivation of an expression for the central charge c, the coefficient of the Weyl tensor squared contribution to the conformal anomaly, in terms of the beta and gamma functions of the theory, of the form

$$c = c_1 + \frac{1}{24}\left(N_V \frac{\beta_g}{g} - \gamma_i {N_\chi}^i\right), \qquad c_1 = \frac{1}{24}\left(3N_V + N_\chi\right), \qquad N_\chi = \sum_i {N_\chi}^i . \tag{1.6}$$

Here ${N_\chi}^i$ denotes the number of chiral fields with anomalous dimension γ_i (We take the anomalous dimension matrix to be diagonal). c_1 is the one-loop result for the central charge. The expression (1.6) was first presented in [38], and is based on the two-loop calculations of [39]. Our all-order derivation presented here relies on the fact that on curved superspace, the topological Ward identity has an additional one-loop anomaly of the form

$$\left(\int d^6z\, \frac{\delta}{\delta\lambda} - \int d^6\bar{z}\, \frac{\delta}{\delta\bar{\lambda}}\right)\Gamma' = \frac{\mathcal{C}_1}{4}\left(\int d^6z\, \frac{1}{\lambda'}\mathrm{tr}(W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}) - \int d^6\bar{z}\, \frac{1}{\bar{\lambda}'}\mathrm{tr}(\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}})\right), \tag{1.7}$$

where $W_{\alpha\beta\gamma}$ is the superspace Weyl density. Γ' is the vertex functional in which a suitable local counterterm has been added to Γ in (1.2) such as to cancel the r.h.s. of (1.2). Note again that the anomaly (1.7) vanishes in the constant coupling limit. We calculate the coefficient $\mathcal{C}_1$ to one loop in the component decomposition of [22], and show how it gives rise to the desired result for c in the same renormalisation scheme as used for the derivation of the NSVZ β function. Moreover we extend our result to theories with matter by considering an off-shell version of the Konishi anomaly on curved space background. The essential point of our derivation is again - as in [7] - that with a suitable local counterterm, the anomaly in (1.7) may be shifted from the topological Ward identity to the superconformal Ward identity and thus to the Callan-Symanzik equation.

The coefficient of the Euler central charge a is inherently more difficult to derive in the approach presented here, since it contains terms non-linear in the anomalous dimension, of the form $\mathrm{tr}(\gamma\gamma)$ and $\mathrm{tr}(\gamma\gamma\gamma)$. We expect to return to a derivation of a in the future. At least already at the present stage we are able to show that there cannot be any terms of the form $\mathrm{tr}(\gamma)$ contributing to a. Moreover our results are consistent with the expected factor of the $\beta(g)/g$ contribution to a.

The outline of this paper is as follows. We begin in section 2 with a brief summary of the results of [7] relevant for our analysis, the implications of local chiral couplings in supersymmetric Yang-Mills theory and the associated internal anomalies. Then we identify the constraints leading to the particular renormalisation scheme which gives rise to the NSVZ β-function. Moreover we extend the analysis to gauge theories with matter. In section 3 we find expressions for the central charge c by considering the external anomalies. Firstly, the gauge field contribution to the new topological anomaly is calculated and is shown to give rise to the $\beta(g)/g$ contribution to c. Next we include the matter contributions by considering the Konishi anomaly in an off-shell approach for a curved superspace background. Finally in section 4 we conclude with comments on the implications for the Euler central charge a and an outlook on future directions. The appendix contains the necessary one-loop triangle diagram computations.

2. Local chiral couplings — internal anomalies

We begin by giving a summary of the results in [7, 9] which show that in a pure gauge theory, a new topological anomaly appears when the couplings are allowed to be space-time dependent. We then show how the NSVZ beta function arises in this approach for a particular renormalisation scheme.

2.1 Topological anomaly in pure gauge theory

The starting point is pure $\mathcal{N} = 1$ SYM with gauge group G, whose classical action is, in the conventions of [7],

$$S_{\text{constant coupling}}[V] = -\frac{1}{4g^2}\int d^6z\,\mathrm{tr}(W^\alpha W_\alpha) - \frac{1}{4g^2}\int d^6\bar{z}\,\mathrm{tr}(\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}), \tag{2.1}$$

with

$$W_\alpha = \frac{1}{8}\bar{D}^2(e^{-2V}D_\alpha e^{2V})\,, \qquad \bar{W}_{\dot\alpha} = -\frac{1}{8}D^2(e^{2V}\bar{D}_{\dot\alpha}e^{-2V})\,. \tag{2.2}$$

Next the gauge coupling g is promoted to a local chiral and antichiral superfield, $\lambda(z)$ and $\bar\lambda(\bar z)$ respectively, such that the action becomes

$$\mathcal{S}[V] = -\frac{1}{2}\int d^6z\,\lambda(z)\mathrm{tr}(W^\alpha W_\alpha) - \frac{1}{2}\int d^6\bar{z}\,\bar\lambda(\bar z)\mathrm{tr}(\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha})\,. \tag{2.3}$$

As discussed in [6], the vector superfield V may *not* be fixed to the Wess-Zumino gauge if manifest supersymmetry is to be preserved in the presence of local couplings. In the case where the Wess-Zumino gauge is fixed, the supersymmetry algebra closes only up to gauge transformations and hence is not linearly realised.

In order to be able to make recourse to perturbation theory, there has to be a well-defined free field limit. For the action (2.3), there are two ways to proceed: If the lowest components of $\lambda(z)$ and $\bar\lambda(\bar z)$ are taken to coincide with the local coupling and local theta angle by virtue of

$$\lambda(z)|_\theta + \bar\lambda(\bar z)|_{\bar\theta} = \frac{1}{2g^2(x)}\,, \qquad \lambda(z)|_\theta - \bar\lambda(\bar z)|_{\bar\theta} = -\frac{i}{16\pi^2}\Theta(x)\,, \tag{2.4}$$

then a rescaling of V by

$$V \to (\lambda + \bar\lambda)^{-1/2}V\,, \tag{2.5}$$

leads to a well-defined free field limit of the action (2.3). It would be very interesting to study the renormalisation behaviour of the action with this normalization.

Here, however, we follow another approach for calculational simplicity, first used in [7]. In this approach a well-defined free field limit is obtained by shifting the lowest components of the superfields by a constant such that

$$\lambda'(z) = \lambda(z) + \frac{1}{2g^2}\,, \qquad \bar\lambda'(\bar z) = \bar\lambda(\bar z) + \frac{1}{2g^2}. \tag{2.6}$$

The new action with these couplings reads

$$S[V] = -\frac{1}{2}\int d^6z\,\lambda'(z)\mathrm{tr}(W^\alpha W_\alpha) - \frac{1}{2}\int d^6\bar{z}\,\bar\lambda'(\bar z)\mathrm{tr}(\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha})\,. \tag{2.7}$$

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From this classical action a perturbative expansion for the associated quantum theory is obtained as follows. Varying the vertex functional corresponding to (2.7) with respect to λ or $\bar{\lambda}$ gives rise to well-defined operator insertions, which in turn correspond to the vertices of the perturbation expansion. Moreover the action (2.7) has a well-defined free field limit, which is obtained by setting $\lambda = 0, \bar{\lambda} = 0$. This allows for an unambiguous definition of the free propagator but with an additional factor of g^2 in its definition as compared to the standard constant coupling perturbative approach. Kraus and collaborators show that the perturbative expansion obtained in this way — vertices from functional differentiation and modified propagators — satisfies

$$N_g = 2(N_\lambda + N_{\bar{\lambda}}) + 2(l-1) + N_V\,, \tag{2.8}$$

where N_g denotes the power of the constant coupling g in a Feynman graph, N_λ and $N_{\bar{\lambda}}$ count the operator insertions obtained by varying with respect to λ or $\bar{\lambda}$, respectively, and N_V the number of external superfield legs. l is the number of loops. The relation (2.8) ensures that the loop expansion is a power series.

For the action (2.7) there are two important classical Ward identities which become anomalous upon quantisation. These arise due to a new global symmetry associated with the local couplings. The relevant symmetry transformation consists of shifting the local couplings by a complex constant ω,

$$\lambda(z) \to \lambda(z) + \omega\,, \qquad \bar{\lambda}(\bar{z}) \to \bar{\lambda}(\bar{z}) + \bar{\omega}\,. \tag{2.9}$$

If we define the following operators

$$\Delta^{\pm} \equiv \int d^6 z\, \frac{\delta}{\delta\lambda(z)} \pm \int d^6 \bar{z}\, \frac{\delta}{\delta\bar{\lambda}(\bar{z})}\,, \tag{2.10}$$

then it is easily verified that the classical action $S[V]$ satisfies the *shift equation*, induced by the real part of ω,

$$\Delta^+ S = -g^3 \partial_g S\,. \tag{2.11}$$

This identity is crucial for establishing the equivalence between the perturbation expansion described above and the standard one.

In addition, there is also the *Pontryagin identity* arising from the imaginary part of ω,

$$\Delta^- S = -\frac{1}{2}\int d^6 z\, \mathrm{tr}(W^\alpha W_\alpha) + \frac{1}{2}\int d^6 \bar{z}\, \mathrm{tr}(\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}) = 0\,. \tag{2.12}$$

The r.h.s. vanishes on flat space since it is an integral over a topological density.

The crucial result of [7] is that in the quantised theory, the Pontryagin identity becomes anomalous. At one loop, the anomaly is given by

$$\Delta^- \Gamma = \frac{\mathcal{A}_1}{4}\left(\int d^6 z\, \frac{1}{\lambda'(z)} \mathrm{tr}(W^\alpha W_\alpha) - \int d^6 \bar{z}\, \frac{1}{\bar{\lambda}'(\bar{z})} \mathrm{tr}(\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}})\right), \tag{2.13}$$

with Γ the BPHZ vertex functional. (Capital calligraphic letters denote anomaly coefficients and the number 1 denotes 1-loop.) The anomaly coefficient $\mathcal{A}_1$ was calculated in [6] and is given by

$$\mathcal{A}_1 = \frac{1}{8\pi^2} C_2(G)\,, \tag{2.14}$$

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where the group theory factor is $(T^A T^A)_R = C(R)\mathbf{1}$ for a representation R of the gauge group, and $R = G$ is the adjoint representation. For $G = \mathrm{SU}(N_c)$ we have

$$\mathcal{A}_1 = \frac{N_c}{8\pi^2}\,. \tag{2.15}$$

It is essential to note that the local couplings enter the anomaly in (2.13) in the form $1/\lambda'$, $1/\bar{\lambda}'$. This is consistent with the fact that when taking the constant coupling limit, the integrand of (2.13) reduces to a component form that contains a factor of g^2 as in (1.5) (see [6]).

2.2 Shifting the anomaly

We have seen that the Pontryagin equation has an anomaly given by (2.13). However by using the freedom of adding local counterterms, $\Gamma^{ct}(V)$, to the quantum action, it is possible to move the anomaly to the shift equation (2.11). This is analogous to the situation for the chiral anomaly, where one has the freedom to shift the anomaly between the axial and vector Ward identities. The specific counterterm chosen in [7], which shifts the anomaly from (2.13) to (2.11), is

$$\begin{aligned}\Gamma^{ct}(V) &= \\ &= -\frac{\mathcal{A}_1}{4}\left(\int d^6z\,[\ln(2\lambda'(z)) + \ln g^2]\mathrm{tr}(W^\alpha W_\alpha) + \int d^6\bar{z}\,[\ln(2\bar{\lambda}'(\bar{z})) + \ln g^2]\,\mathrm{tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})\right).\end{aligned} \tag{2.16}$$

The $\ln g^2$ terms in this expression are necessary to ensure that this counterterm vanishes in the constant coupling limit $\lambda = 0, \bar{\lambda} = 0$, such that the constant coupling perturbation expansion remains a power series. The action

$$\Gamma' = \Gamma + \Gamma^{ct}(V) \tag{2.17}$$

satisfies the Pontryagin identity

$$\Delta^-\Gamma' = 0\,. \tag{2.18}$$

However the shift equation now becomes anomalous,

$$\Delta^+\Gamma' = -g^3\partial_g\Gamma' - \frac{\mathcal{A}_1 g^2}{2}\left(\int d^6z\;\mathrm{tr}(W^\alpha W_\alpha) + \int d^6\bar{z}\;\mathrm{tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})\right). \tag{2.19}$$

The anomaly term is just the classical SYM action being acted upon by the operator Δ^+, such that at one loop, (2.19) may be written as

$$\Delta^+\Gamma' = -g^3\partial_g\Gamma' + \mathcal{A}_1 g^2\Delta^+ S\,. \tag{2.20}$$

This result is scheme independent.

We now proceed by chosing a particular scheme, which we show to coincide with the NSVZ scheme. A particular scheme is obtained by assuming that (2.18) and (2.20) are valid to *all* orders in perturbation theory, which corresponds to replacing (2.20) by the full quantum equation

$$\Delta^+\Gamma' = -g^3\partial_g\Gamma' + \mathcal{A}_1 g^2\Delta^+\Gamma'\,. \tag{2.21}$$

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This scheme choice requires in particular that the Pontryagin anomaly we discussed above is exhausted at one-loop in this particular scheme.[2]

It is straightforward to see that the scheme defined by passing from (2.20) to (2.21) is indeed the NSVZ scheme in which the beta function takes the form derived by NSVZ in [14]. Indeed, a rearrangement of (2.21) gives

$$\Delta^{+}\Gamma' = -\frac{g^3}{1-\mathcal{A}_1 g^2}\partial_g \Gamma' \,. \tag{2.22}$$

This has already a form reminiscent of the NSVZ β-function. For an exact identification we now establish the connection to scale transformations of the vertex functional as expressed by a renormalisation group flow.

As discussed in [7], when (2.18) holds to all orders, the local coupling λ (or $\bar{\lambda}$) is not renormalised beyond one loop and remains holomorphic (or antiholomophic) in the quantised theory. This is due to the consistency condition

$$\left[\Delta^{-}\,,\ \mu\frac{\partial}{\partial\mu}\right]\Gamma' = 0\,, \tag{2.23}$$

with Δ^- defined in (2.10) and $[\,,\,]$ the commutator. Therefore the Callan-Symanzik equation reads

$$\left(\mu\frac{\partial}{\partial\mu} + \mathcal{B}_1\Delta^{+}\right)\Gamma' = 0\,, \qquad \mathcal{B}_1 = \frac{3}{16\pi^2}C_2(G)\,, \tag{2.24}$$

with Δ^+ defined in (2.10) and $\mathcal{B}_1$ the standard one-loop coefficient of the gauge beta function. Inserting (2.22) into (2.24) then gives rise to to the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\partial_g\right)\Gamma' = 0\,, \tag{2.25}$$

with

$$\beta(g) = -\frac{\mathcal{B}_1 g^3}{1-\mathcal{A}_1 g^2} = -\frac{g^3}{16\pi^2}\left(\frac{3C_2(G)}{1-C_2(G)g^2/(8\pi^2)}\right)\,, \tag{2.26}$$

which is precisely the NSVZ β-function.

A further important point is that if (2.18) is valid to all orders in perturbation theory, then a well-defined local operator insertion is obtained by virtue of

$$\frac{\delta}{\delta\lambda(z)}\Gamma' \;=\; -\frac{1}{2}\,\mathrm{tr}\,W^{\alpha}W_{\alpha}(z)\,. \tag{2.27}$$

In standard notation, local insertions of composite operators are often denoted by square brackets, ie. $[W^{\alpha}W_{\alpha}]$, but we omit this here and in the remainder of this paper to simplify the notation.

The Callan-Symanzik equation (2.24) may be interpreted as an anomalous Ward identity for scale transformations. Scale transformation are a subgroup of superconformal transformations, and together with the local equation (2.27), (2.24) implies that the su-

[2]It appears feasible to prove one-loop exactness of the Pontryagin anomaly by an argument based on the results of [12] together with supersymmetry. However finite renormalisations at higher order are always possible.

perconformal Ward identity consistent with (2.24) is given by

$$\bar{D}^{\dot\alpha} T_{\alpha\dot\alpha} \,=\, D_\alpha T\,, \qquad T = -\frac{1}{6}\mathcal{B}_1 \text{ tr } W^\beta W_\beta\,. \tag{2.28}$$

Here $T_{\alpha\dot\alpha}$ is the supercurrent from which all currents of the superconformal group may be obtained. Note that for the theory with local couplings considered here, the supercurrent has contributions which involve derivatives of the couplings. Moreover the superconformal anomaly T is one-loop.

2.3 Curved superspace background

We proceed by deriving a local version of the Callan-Symanzik equation (2.25). For this purpose we couple the quantised local coupling theory to a classical curved superspace background. As in [29, 30], the curved superspace involves a chiral compensator ϕ and a real Weyl invariant superfield $H^{\alpha\dot\alpha}$, such that local quantum insertions of the supercurrent $T_{\alpha\dot\alpha}$ and of the superconformal anomaly T are given by

$$\bar{\mathcal{D}}^{\dot\alpha} T_{\alpha\dot\alpha} \,=\, \mathcal{D}_\alpha T\,, \qquad T_{\alpha\dot\alpha} = \frac{\delta}{\delta H^{\alpha\dot\alpha}}\Gamma'\,, \qquad T \,=\, \frac{1}{3}\,\phi\frac{\delta}{\delta\phi}\Gamma'\,. \tag{2.29}$$

It is important to note that in the case of constant couplings, T as given by (2.29) has higher-order quantum corrections. It is possible to redefine the supercurrent $T_{\alpha\dot\alpha}$ such that T is one-loop, but then the coupling to curved superspace as given by (2.29) is inconsistent (see [40 − 42, 30]).

However for the local coupling theory considered here, the curved superspace background given by (2.29) is consistent with T being one-loop, as we now show. This is possible essentially since the supercurrent is modified by additional terms involving derivatives of the couplings.

When coupled to a curved superspace background as in [29, 30], the classical action (2.7) becomes

$$S[V] = -\frac{1}{2}\int d^6 z\,\phi^3\,\lambda'(z)\text{tr}(W^\alpha W_\alpha) - \frac{1}{2}\int d^6\bar z\,\bar\phi^3\,\bar\lambda'(\bar z)\,\text{tr}(\bar W_{\dot\alpha}\bar W^{\dot\alpha})\,. \tag{2.30}$$

The topological anomaly discussed in section 2.1 is present also on the curved superspace background. A quantum action Γ' satisfying the topological Ward identity $\Delta^-\Gamma' \,=\, 0$ as in (2.18) is obtained from the quantum action Γ corresponding to the classical action (2.30) by adding a suitable local counterterm, $\Gamma' \,=\, \Gamma \,+\, \Gamma^{ct}(V)$ given by

$$\begin{aligned}\Gamma^{ct}(V) \,=\, &-\frac{\mathcal{A}_1}{4}\int d^6 z\,\phi^3[\ln(2\lambda'(z)) + \ln g^2]\text{tr}(W^\alpha W_\alpha) - \\ &-\frac{\mathcal{A}_1}{4}\int d^6 \bar z\,\bar\phi^3[\ln(2\bar\lambda'(\bar z)) + \ln g^2]\text{tr}(\bar W_{\dot\alpha}\bar W^{\dot\alpha})\,,\end{aligned} \tag{2.31}$$

which is the curved superspace analogue of (2.16). This counterterm is Weyl invariant.

The Ward identity $\Delta^-\Gamma' \,=\, 0$ is crucial for our construction. It implies that a local operator insertion is obtained by virtue of

$$\frac{\delta}{\delta\lambda}\Gamma' \,=\, \phi^3 \text{ tr } W^\alpha W_\alpha\,. \tag{2.32}$$

Then the consistency condition

$$\left[\phi\frac{\delta}{\delta\phi}\,,\,\frac{\delta}{\delta\lambda}\right]\Gamma' \;=\; 0 \tag{2.33}$$

implies that

$$T \;=\; \frac{1}{3}\,\phi\frac{\delta}{\delta\phi}\,\Gamma' \tag{2.34}$$

is one-loop. Note that $T_{\alpha\dot\alpha}$, the insertion of the supercurrent given by (2.29), contains derivatives of the local couplings and is therefore different from the supercurrent in the standard constant coupling theory.

By dimensional analysis, the quantum vertex functional Γ' satisfies the scale relation

$$\mu\frac{\partial}{\partial\mu}\Gamma' + \left(\int d^6z\,\phi\frac{\delta}{\delta\phi} + \int d^6\bar z\,\bar\phi\frac{\delta}{\delta\bar\phi}\right)\Gamma' \;=\; 0\,. \tag{2.35}$$

The consistency condition (2.33) implies that the Callan-Symanzik equation of the flat space case (2.24) is also valid for the theory coupled to curved superspace,

$$\left(\mu\frac{\partial}{\partial\mu} + \mathcal{B}_1\Delta^+\right)\Gamma' \;=0\,,\qquad \mathcal{B}_1 = \frac{3}{16\pi^2}C_2(G)\,. \tag{2.36}$$

A local Callan-Symanzik equation consistent with (2.36) and (2.35) is given by

$$\phi\frac{\delta}{\delta\phi}\Gamma' \;=\; \mathcal{B}_1\,\frac{\delta}{\delta\lambda}\Gamma'\,. \tag{2.37}$$

The superconformal anomaly reads

$$\bar{\mathcal{D}}^{\dot\alpha}T_{\alpha\dot\alpha} \;=\; \mathcal{D}_\alpha T\;,\qquad T \;=\; -\frac{\mathcal{B}_1}{6}\,\phi^3\,\mathrm{tr}W^\alpha W_\alpha\,. \tag{2.38}$$

This is the curved superspace analogue of (2.28).

2.4 Theories with matter

Lets us now consider the action of a gauge theory with matter,

$$S = S_{\rm gauge} + S_{\rm matter}\,, \tag{2.39}$$

where the gauge part of the action is as in (2.7) above, and the matter part contains n chiral fields Φ^i transforming in the representations R_i of the gauge group G. On a curved space background, the matter action is given by (see [29, 30] for superspace notation)

$$S_{\rm matter} = \frac{1}{4}\int d^8z\,\tilde E\bar\Phi_i e^{2V}\Phi^i\,. \tag{2.40}$$

$\tilde E$ is the appropriate curved superspace integration measure and i is the flavour index. We suppress colour indices for notational simplicity. We first restrict to the case where there is no superpotential, and include it in a second step below.

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The new vertex functional corresponding to (2.39) still satisfies the Pontryagin equation (2.18) with the same anomaly as before. Therefore the shift equation (2.21) is preserved. This implies in particular that the denominator for the NSVZ β-function is unchanged as expected.

We now generalise the local Callan-Symanzik equation (2.37) to the action (2.39) which includes chiral matter. Our starting point is the result of [35 – 37], based on an earlier result in [34], according to which on flat space, the supercurrent anomaly for the action (2.39) may be written as

$$\bar{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = -\frac{1}{3} D_\alpha \left(\frac{{\mathcal{B}_1}'}{2} \mathrm{tr}(W^\beta W_\beta) + \frac{1}{4} \bar{D}^2 \sum_{i=1}^{n} \gamma_i \bar{\Phi}_i e^{2V} \Phi^i \right), \tag{2.41}$$

where the coefficient of the gauge anomaly is one-loop and γ_i is the anomalous dimension[3] of the field Φ^i. The coefficient $\mathcal{B}_1'$ is given by

$$\mathcal{B}_1' = \frac{1}{16\pi^2} \left(3C_2(G) - \sum_{i=1}^{n} T(R_i) \right), \tag{2.42}$$

where $T(R_i)$ is the Dynkin index in the representation R_i of G, $\mathrm{tr}(T^A T^B)_{R_i} = T(R_i)\delta^{AB}$.

We now generalise (2.41) in two ways: We consider the off-shell case and we couple the quantised theory to a classical supergravity background as in section 2.4. Then the supercurrent anomaly is given by

$$\bar{\mathcal{D}}^{\dot{\alpha}} \frac{\delta}{\delta H^{\alpha\dot{\alpha}}} \Gamma' = \frac{1}{3} \mathcal{D}_\alpha \left[\left(\phi \frac{\delta}{\delta\phi} - \Phi^i \frac{\delta}{\delta\Phi^i} \right) \Gamma' \right], \tag{2.43}$$

where the r.h.s. is the transformation of the action under the super Weyl transformation given by $\delta\phi = \sigma\phi$, $\delta\Phi^i = -\sigma\Phi^i$ with $\sigma(z)$ the (chiral) Weyl transformation parameter.[4] (2.43) generalises (2.29) to the matter case. Γ' is the vertex functional obtained from the classical action (2.39), together with the necessary counterterm (2.31) to guarantee $\Delta^- \Gamma' = 0$ with Δ^- as in (2.10). As discussed in section 2.4, the supercurrent in the local coupling theory on curved superspace has a one-loop gauge anomaly. Therefore instead of the flat space equation (2.41) we may now write

$$\left(\phi \frac{\delta}{\delta\phi} - \Phi^i \frac{\delta}{\delta\Phi^i} \right) \Gamma' = -\frac{{\mathcal{B}_1}'}{2} \phi^3 \, \mathrm{tr}\,(W^\alpha W_\alpha) + \frac{1}{4} \phi^3 (\bar{\mathcal{D}}^2 + R) \sum_{i=1}^{n} \gamma_i \bar{\Phi}_i e^{2V} \Phi^i , \tag{2.44}$$

which generalises (2.34) and (2.38). The anomalous Weyl transformation (2.44) is the starting point for deriving the local Callan-Symanzik equation in the presence of matter fields. It should be kept in mind that there are also contributions from the gravitational anomalies to (2.44). These are discussed separately in section 3.

[3] As in [37], we assume that the anomalous dimension matrix is diagonal, ${\gamma^i}_j = \gamma_{(i)} {\delta^i}_j$. Note also that the anomalous dimension of Φ^i used here is half the value of the mass anomalous dimension used in [36, 37, 22]. We use the superspace conventions of [37].

[4] For a detailed analysis of the transformation properties of quantum actions and of the quantum supercurrent under superconformal transformations, see [31 – 33].

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Next we combine (2.44) with the Konishi anomaly. The Konishi anomaly corresponds to an anomaly in an axial symmetry under which the matter fields transform as $\delta\Phi^i = iv\Phi^i$, $v \in \mathbb{R}$. The classical action is invariant under this symmetry,

$$\left[\int d^6z\,\Phi^i\frac{\delta}{\delta\Phi^i} \;-\; \int d^6\bar{z}\,\bar{\Phi}^i\frac{\delta}{\delta\bar{\Phi}^i}\right] S \;=\; 0\,. \tag{2.45}$$

The current associated with this symmetry is precisely the Konishi current: The local version of (2.45) is

$$\Phi^i\frac{\delta}{\delta\Phi^i}S = -\frac{1}{4}\phi^3(\bar{\mathcal{D}}^2+R)(\bar{\Phi}_i e^{2V}\Phi^i)\,. \tag{2.46}$$

In the quantised theory, the Konishi current has an anomaly [43] which takes the off-shell form[5]

$$\Phi^i\frac{\delta}{\delta\Phi^i}\Gamma' = \frac{1}{4}\phi^3(\bar{\mathcal{D}}^2+R)(\bar{\Phi}_i e^{2V}\Phi^i) \;-\; \frac{1}{16\pi^2}\sum_{i=1}^{n}T(R_i)\,\phi^3\mathrm{tr}(W^\alpha W_\alpha)\,, \tag{2.47}$$

or

$$\gamma_i\Phi^i\frac{\delta}{\delta\Phi^i}\Gamma' = \gamma_i\frac{1}{4}\phi^3(\bar{\mathcal{D}}^2+R)(\bar{\Phi}_i e^{2V}\Phi^i) \;-\; \frac{1}{16\pi^2}\sum_{i=1}^{n}\gamma_i T(R_i)\,\phi^3\mathrm{tr}(W^\alpha W_\alpha)\,. \tag{2.48}$$

Combining (2.48) and (2.44) we obtain the local Callan-Symanzik equation

$$\left[\phi\frac{\delta}{\delta\phi} - (1-\gamma_i)\Phi^i\frac{\delta}{\delta\Phi^i}\right]\Gamma' \;=\; \left(\mathcal{B}'_1 + \frac{2}{16\pi^2}\sum_{i=1}^{n}\gamma_i T(R_i)\right)\frac{\delta}{\delta\lambda}\Gamma'\,. \tag{2.49}$$

Using

$$\mu\frac{\partial}{\partial\mu}\Gamma' \;+\; \left(\int d^6z\,\left(\phi\frac{\delta}{\delta\phi} - \Phi^i\frac{\delta}{\delta\Phi^i}\right) + \text{c.c.}\right)\Gamma' \;=\; 0\,, \tag{2.50}$$

which generalises (2.35), and the definition of Δ^+ given in (2.10), we have

$$\left[\mu\frac{\partial}{\partial\mu} + \left(\mathcal{B}'_1 + \frac{2}{16\pi^2}\sum_{i=1}^{n}\gamma_i\right)\Delta^+ - \gamma_i\mathcal{N}_i\right]\Gamma' = 0\,, \tag{2.51}$$

with

$$\mathcal{N}_i \equiv \int d^6z\,\left(\Phi^i\frac{\delta}{\delta\Phi^i} + \text{c.c.}\right). \tag{2.52}$$

By virtue of the shift equation (2.22) we obtain the standard Callan-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \left(\mathcal{B}'_1 + \frac{2}{16\pi^2}\sum_{i=1}^{n}\gamma_i\right)\left(\frac{-g^3}{1-\mathcal{A}_1 g^2}\right)\partial_g - \gamma_i\mathcal{N}_i\right]\Gamma' = 0\,. \tag{2.53}$$

Thus with $\mathcal{B}'_1$ given by (2.42) we find for the β-function that

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{3C_2(G) - \sum_{i=1}^{n}(T(R_i) - 2\gamma_i T(R_i))}{1 - C_2(G)g^2/8\pi^2}\right), \tag{2.54}$$

which coincides with the NSVZ beta function.

[5] again in the conventions used in [37] for the Konishi anomaly.

It is straightforward to include a superpotential with local chiral supercoupling into this discussion. In this case the matter part of the action is given by

$$S_{\text{matter}} = \frac{1}{4}\int d^8z\,\tilde{E}\,\bar{\Phi}_i e^{2V}\Phi^i + \frac{1}{3!}\int d^6z\,\phi^3 Y_{ijk}\Phi^i\Phi^j\Phi^k + \frac{1}{3!}\int d^6\bar{z}\,\bar{\phi}^3\bar{Y}^{ijk}\bar{\Phi}_i\bar{\Phi}_j\bar{\Phi}_k\,. \quad (2.55)$$

In this theory the matter beta function satisfies

$$\beta^{ijk}(Y) = 3\gamma^{(i}_{\ m}Y^{jk)m}\,. \quad (2.56)$$

As before the classical action is invariant under the Konishi symmetry, under which the matter fields and the local matter couplings transform as

$$\begin{aligned} &\Phi^i \to e^{iv}\Phi^i\,, \qquad Y_{ijk} \to e^{-3iv}Y_{ijk}\,,\\ &\bar{\Phi}_i \to e^{-iv}\bar{\Phi}_i,\ \bar{Y}^{ijk} \to e^{3iv}\bar{Y}^{ijk}\,, \qquad v\in\mathbb{R}\,. \end{aligned} \quad (2.57)$$

The superpotential remains invariant under this transformation, as well as the matter kinetic term. Note also that this symmetry differs from R symmetry since it leaves the chiral compensator invariant. The symmetry (2.57) leads to the Ward identity

$$\left(\int d^6z\left[3Y^{ijk}\frac{\delta}{\delta Y^{ijk}} - \Phi^i\frac{\delta}{\delta\Phi^i}\right] - c.c.\right)S = 0\,. \quad (2.58)$$

In the quantised theory the Konishi anomaly gives rise to

$$\left(\Phi^i\frac{\delta}{\delta\Phi^j} - 3Y^{ijk}\frac{\delta}{\delta Y^{ijk}}\right)\Gamma' = \frac{1}{4}\phi^3(\bar{\mathcal{D}}^2+R)(\bar{\Phi}_i e^{2V}\Phi^i) - \frac{1}{16\pi^2}\sum_{i=1}^{n}T(R_i)\phi^3\mathrm{tr}(W^\alpha W_\alpha)\,. \quad (2.59)$$

Note that due to the variation with respect to the supercoupling on the l.h.s., there is no contribution from the superpotential on the r.h.s. of (2.59). The absence of an anomaly involving the superpotential from this equation may be seen as follows: As discussed for instance in [42], for renormalisable massless SUSY field theories there is no superpotential contribution to the conformal Ward identity (2.41). This is a consequence of R symmetry and leads in particular to the relation (2.56) between the matter beta and gamma functions. Therefore there is no superpotential contribution to the r.h.s. of (2.44). The same argument implies that there is no superpotential contribution to the r.h.s. of (2.59).[6]

By adding (2.59) to the integrated off-shell super Weyl identity (2.44) we obtain the Callan-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\partial_g + \left(\int d^6z\,\beta^{ijk}\frac{\delta}{\delta Y^{ijk}} + \int d^6\bar{z}\,\bar{\beta}_{ijk}\frac{\delta}{\delta\bar{Y}_{ijk}}\right) - \gamma_i\mathcal{N}_i\right]\Gamma' = 0\,. \quad (2.60)$$

with the same NSVZ gauge β-function as before. There are additional contributions from the gravitational anomalies to the r.h.s. of (2.60) which we discuss next.

[6] We are grateful to E. Sokatchev for a discussion on this point. See [44] for cases where mixing occurs.

3. External anomalies — central functions

In addition to the internal anomalies discussed above, there will also be external anomalies involving the super Euler density and the square of the supergravity Weyl tensor. These external anomalies appear both in the topological and in the conformal Ward identities. We calculate the coefficient of the gravitational anomalies in the topological Ward identity (2.18) to one-loop order and show how an anomaly shift similar to the one performed in section 2 above allows us to derive an all-order expression for the central charge c (the coefficient of the Weyl tensor squared) in a particular well-defined renormalisation scheme. This expression coincides with the one found in [22] based on a two-loop result of Jack [39].

3.1 Gauge field contribution

As before we first consider pure gauge theory. In analogy to the gauge anomaly in the topological Ward identity we expect a gravitational anomaly of the form

$$\Delta^{-}\Gamma' = \frac{\mathcal{F} N_V}{24\pi^2}\, W^2 - \frac{\mathcal{G} N_V}{24\pi^2}\, E^2, \tag{3.1}$$

where

$$\begin{aligned} W^2 &= \left(\int d^6 z\, \phi^3 \lambda'^{-1}(z) W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} - \int d^6 \bar{z}\, \bar{\phi}^3 \bar{\lambda}'^{-1}(\bar{z}) \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} \right), \\ E^2 &= \int d^6 z\, \phi^3 \lambda'^{-1}(z) (W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + (\bar{\mathcal{D}}^2 + R)[G^a G_a + 2R\bar{R}]) - \\ &\quad - \int d^6 \bar{z}\, \bar{\phi}^3 \bar{\lambda}'^{-1}(\bar{z}) (\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} + (\mathcal{D}^2 + \bar{R})[G^a G_a + 2R\bar{R}])\,. \end{aligned} \tag{3.2}$$

Here $W_{\alpha\beta\gamma}$ is the super-Weyl tensor and $(W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + (\bar{\mathcal{D}}^2 + R)[G^a G_a + 2R\bar{R}])$ is the chiral projection of the the super Euler density [29, 30]. By performing the one-loop calculation of appendix A.2, we find for the coefficients $\mathcal{F}$, $\mathcal{G}$ to one-loop order:

$$\mathcal{F}_1 = -\frac{1}{32}\,, \qquad \mathcal{G}_1 = -\frac{3}{16}\,. \tag{3.3}$$

A summary of the one-loop calculation of appendix A.2 is as follows: we decompose (3.1) into components and perform the required triangle diagram computations for the topological current, in analogy to the gauge anomaly computation outlined in (1.4), (1.5). In particular it is necessary to calculate one-loop three point functions which contain the the topological current (1.4) and two copies of either the R current or the energy-momentum tensor — in the subsequent we assume that an appropriate scheme may be chosen such that there are no higher order contributions to $\mathcal{F}$. This gives a result for c consistent with expectations. As far as $\mathcal{G}$ is concerned, we work with its lowest-order value here and leave an investigation of its higher order contributions to future investigations.

In analogy to the discussion of section 2.2, the anomaly (3.1) may be shifted to the 'shift equation' (2.21) by adding an appropriate counterterm to the action. This counterterm reads

$$\begin{aligned}
\Gamma^{ct}(W,E) = &-\frac{\mathcal{F}_1 N_V}{24\pi^2}\int d^6z\,\phi^3[\ln(\lambda'(z))+\ln 2g^2]W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} - \\
&-\frac{\mathcal{F}_1 N_V}{24\pi^2}\int d^6\bar{z}\,\bar{\phi}^3[\ln(\bar{\lambda}'(\bar{z}))+\ln 2g^2]\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} + \\
&+\frac{\mathcal{G}_1 N_V}{24\pi^2}\int d^6z\,\phi^3\ln(\lambda'(z))(W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}+(\bar{\mathcal{D}}^2+R)[G^aG_a+2R\bar{R}]) + \\
&+\frac{\mathcal{G}_1 N_V}{24\pi^2}\int d^6\bar{z}\,\bar{\phi}^3\ln(\bar{\lambda}'(\bar{z}))(\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}+(\mathcal{D}^2+\bar{R})[G^aG_a+2R\bar{R}])\,. \qquad (3.4)
\end{aligned}$$

The $\ln 2g^2$ is necessary as before to obtain a well-defined constant coupling limit. Note since the Euler density is topological, the terms in which it is multiplied by the constant $\ln 2g^2$ actually vanish, and such terms are absent from the counterterm above. For

$$\Gamma'' = \Gamma' + \Gamma^{ct}(W,E) \qquad (3.5)$$

the quantum Pontryagin equation is anomaly free,

$$\Delta^-\Gamma'' = 0\,, \qquad (3.6)$$

but the shift equation (2.21) becomes anomalous as in the case of the internal anomaly discussed in section 2. Let us derive the exact form of this external anomaly.

For this purpose we first determine how the topological gravitational anomaly contributes to the conformal anomaly. We begin by considering the contributions of the gravitational anomaly to the supercurrent divergence,

$$\begin{aligned}
\bar{\mathcal{D}}^{\dot{\alpha}}T_{\alpha\dot{\alpha}} &= \mathcal{D}_\alpha(T_{\text{internal}}+T_{\text{external}})\,, && (3.7)\\
T_{\text{internal}} &= -\frac{1}{6}\mathcal{B}_1\,\phi^3W^\beta W_\beta\,, \qquad \mathcal{B}_1=\frac{3}{16\pi^2}C_2(G)\,, && (3.8)\\
T_{\text{external}} &= -\frac{c_1-a_1}{48\pi^2}\,\phi^3W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \\
&\quad +\frac{a_1}{48\pi^2}\,\phi^3(\bar{\mathcal{D}}^2+R)(G^aG_a+2R\bar{R})\,. && (3.9)
\end{aligned}$$

Just as the internal anomaly contributes with the one-loop coefficient $\mathcal{B}_1$ to the superconformal anomaly (see (2.33) and (2.38)), the gravitational anomaly contributes with one-loop coefficients c_1 and a_1 given by[7]

$$c_1 = \frac{1}{24}(3N_V+N_\chi)\,, \qquad a_1=\frac{1}{24}(9N_V+N_\chi)\,. \qquad (3.10)$$

This result for c_1, a_1 was first obtained in [45]. (3.9) implies that in the presence of the supergravity background, the local Callan-Symanzik equation (2.37) reads

$$\begin{aligned}
\phi\frac{\delta}{\delta\phi}\Gamma'' = \mathcal{B}_1\frac{\delta}{\delta\lambda}\Gamma'' &- \frac{c_1-a_1}{16\pi^2}\,\phi^3W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \\
&+\frac{a_1}{16\pi^2}\,\phi^3(\bar{\mathcal{D}}^2+R)[G^aG_a+2R\bar{R}]\,. \qquad (3.11)
\end{aligned}$$

[7]For the pure gauge theory considered here, $N_\chi = 0$ but we include the matter contribution for reference in the next section.

The shift equation (2.22) now becomes

$$\mathcal{B}_1\Delta^+\Gamma'' = \beta(g)\partial_g\Gamma'' + \frac{2\mathcal{F}_1 N_V}{24\pi^2}\left(\frac{\beta(g)}{g}\right)\left(\int d^6z\,\phi^3 W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{c.c.}\right), \tag{3.12}$$

where we use the scheme, given by (2.22), in which $\beta(g)$ is the NSVZ β-function. Note again the absence of the Euler density from the shift equation (3.12), since this is a topological invariant and its integral vanishes. The Callan-Symanzik equation now reads

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\partial_g\right)\Gamma'' = \frac{1}{16\pi^2}\left(c_1 - \frac{4}{3}\mathcal{F}_1 N_V\frac{\beta(g)}{g}\right)\left(\int d^6z\,\phi^3 W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{c.c.}\right). \tag{3.13}$$

From this expression we read off the result for the central charge c to all orders,

$$c(g) = c_1 - \frac{4}{3}N_V\mathcal{F}_1\frac{\beta(g)}{g}. \tag{3.14}$$

Using the one-loop result (3.3) for $\mathcal{F}_1$ we have

$$c(g) = c_1 + \frac{N_V}{24}\frac{\beta(g)}{g}. \tag{3.15}$$

This coincides with the result of [22].

3.2 Matter contribution

Let us now include matter fields into the discussion of the section 3.1 in view of deriving an expression for the central charge c in the presence of matter. In generalisation of (2.44) we have, including the gravitational anomaly,

$$\begin{aligned}\left(\phi\frac{\delta}{\delta\phi} - \Phi^i\frac{\delta}{\delta\Phi^i}\right)\Gamma' = &-\frac{{\mathcal{B}_1}'}{2}\phi^3\,\text{tr}(W^\alpha W_\alpha) + \frac{1}{4}\phi^3(\bar{\mathcal{D}}^2 + R)\sum_{i=1}^{n}\gamma_i\bar{\Phi}_i e^{2V}\Phi_i - \\ &-\frac{c_1 - a_1}{16\pi^2}\phi^3 W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \frac{a_1}{16\pi^2}\phi^3(\bar{\mathcal{D}}^2 + R)(G^{\alpha\dot{\alpha}}G_{\alpha\dot{\alpha}} + 2R\bar{R}),\end{aligned} \tag{3.16}$$

where c_1, a_1 are the one-loop coefficients (3.10) of the gravitational Weyl anomaly. Similarly there are now gravitational anomalies contributing to the Konishi identity as well,

$$\begin{aligned}\Phi^i\frac{\delta}{\delta\Phi^i}\Gamma' = &\frac{1}{4}\phi^3(\bar{\mathcal{D}}^2 + R)(\bar{\Phi}_i e^{2V}\Phi^i) - \frac{1}{16\pi^2}\sum_i T(R_i)\,\phi^3\text{tr}(W^\alpha W_\alpha) - \\ &-\frac{1}{24\pi^2}(\mathcal{H} - \mathcal{I})\phi^3 W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \frac{\mathcal{I}}{24\pi^2}\phi^3(\bar{\mathcal{D}}^2 + R)(G^{\alpha\dot{\alpha}}G_{\alpha\dot{\alpha}} + 2R\bar{R}).\end{aligned} \tag{3.17}$$

We calculate $\mathcal{H}$, $\mathcal{I}$ to one loop in appendix A.3 and find

$$\mathcal{H}_1 = -\frac{1}{16}\sum_i {N_\chi}^i, \qquad \mathcal{I}_1 = 0. \tag{3.18}$$

${N_\chi}^i$ is the number of chiral fields with anomalous dimension γ_i. For example, in SQCD with N_f flavours, there is only one γ and $N_\chi = 2N_f N_c$. The fact the $\mathcal{I}$ vanishes at one loop agrees with the fact that there is no contribution linear in γ to the central charge a, as discuss in section 4 below.

By combining the Weyl identity (3.16) and the Konishi identity (3.17) as before, and using Γ'' defined in (3.5), we obtain

$$\left(\mu\frac{\partial}{\partial\mu} - \gamma_i\mathcal{N}_i\right)\Gamma'' = -(\mathcal{B}'_1 + \frac{2}{16\pi^2}\sum_{i=1}^{n}\gamma_i)\Delta^+\Gamma'' + \\ + \frac{1}{16\pi^2}\left(c_1 - \frac{\gamma_i}{24}N_\chi{}^i\right)\left(\int d^6z\,\phi^3\,W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + c.c.\right). \quad (3.19)$$

Therefore we have

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\partial_g - \gamma_i\mathcal{N}_i\right)\Gamma'' = \frac{1}{16\pi^2}\,c\,\left(\int d^6z\,\phi^3\,W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{c.c.}\right), \quad (3.20)$$

with the central charge

$$c = c_1 + \frac{1}{24}\left(N_V\frac{\beta_g}{g} - \gamma_i N_\chi{}^i\right), \qquad c_1 = \frac{1}{24}(3N_V + \sum_i N_\chi{}^i). \quad (3.21)$$

This all-order expression coincides with the result of [22].[8]

4. Conclusions

In this paper we have given a derivation of the NSVZ beta function and of the central charge c in a well-determined renormalization scheme. Our derivation is based on the topological anomaly which is present when the couplings are allowed to be space-time dependent.

As an outlook we comment on the implications of our results on the central charge a, the coefficient of the Euler density anomaly which is expected to be of relevance for generalizations of the C theorem to four dimensions. In [21], a four-loop expression for a candidate C function $\tilde{a}$ was obtained. Here we chose a particular renormalisation scheme in which the result of [21] reads

$$\tilde{a} \equiv a + \beta^i w_i\,, \qquad \tilde{a} = a_1 - \frac{1}{8}\text{tr}(\gamma\gamma) + \frac{1}{12}\,\text{tr}(\gamma\gamma\gamma) + \frac{1}{4}N_V\frac{\beta(g)}{g} + \frac{1}{3}Y^{ijk}\beta_{ijk}\,. \quad (4.1)$$

Here w_i is a one-form in coupling space which may be identified with the coefficient of a local conformal anomaly involving derivatives of the couplings. For a derivation of (4.1) valid to all orders using the approach presented in this paper, it would be necessary to calculate the coefficients of the new anomalies - for instance of $\mathcal{H}$ and $\mathcal{I}$ in (3.17) - to higher order, which we have not undertaken here. So far our statements are limited to expressions linear in the beta and gamma functions. However the results of this paper are consistent with (4.1) in at least two respects. First, the fact that $\mathcal{I} = 0$ at one loop in (3.17) is in agreement with the fact that there is no contribution of the form $\text{tr}(\gamma)$ (linear in γ without further contributions from the couplings) to $\tilde{a}$ as given by (4.1): A term of the form $\mathcal{I}\text{tr}(\gamma)$ would enter a when adding (3.17) multiplied by γ to (3.16). Secondly, the

[8] Note that the anomalous dimension used in [22] is twice the size of the one used here. At the same time $N_\chi = N_f N_c$ in [22], whereas our conventions give $N_\chi = 2N_f N_c$ for the theories considered in [22].

contribution $\frac{1}{4}N_V\frac{\beta(g)}{g}$ to (4.1) is consistent with $\mathcal{G}=-\frac{3}{16}$ in (3.3), just as $\mathcal{F}=-\frac{1}{32}$ leads to the $\frac{1}{24}N_V\frac{\beta(g)}{g}$ contribution to c as given by (3.21). The relative numerical factor is 4/3 in both cases. However the derivation of c from $\mathcal{F}$ presented here does not apply to a and $\mathcal{G}$ since the Euler density contribution is absent from the Callan-Symanzik equation (3.20). - We intend to return to the question of deriving an all-order expression for a within the local coupling approach in the future.

Finally let us note that local couplings appear naturally within the AdS/CFT correspondence and its generalisations since the couplings appear as boundary values of the supergravity fields. In particular the holographic C theorem of [46], valid for field theories in arbitrary even dimensions, may be interpreted within field theory within the local coupling approach [47 − 49]. We expect that the results of the present paper will allow for a further understanding of the field-theoretical implications of the holographic C theorem.

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A. Anomaly coefficients at one loop

A.1 $\mathcal{N}=1$ SYM in components

For completeness, we give here the super Yang-Mills action in components, which we use in the calculations below. For compatibility with [22], we use Euclidean signature in this appendix. We have

$$S = S_D + S_T\,, \tag{A.1}$$

where S_D is the dynamical part [50]

$$S_D = \int d^4x\,\frac{1}{2g^2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu} + 2\bar\lambda\gamma^\mu D_\mu\lambda]\,, \tag{A.2}$$

and S_T the topological part

$$S_T = -\frac{i}{2}\int d^4x\,\tilde\theta\,\mathrm{tr}[F_{\mu\nu}\tilde F^{\mu\nu} - 2\partial_\mu(\bar\lambda\gamma^\mu\gamma_5\lambda)]\,, \qquad \tilde\theta = \frac{\theta}{8\pi^2}\,. \tag{A.3}$$

A.2 Calculating the coefficients of the gravitational topological anomaly to one loop

To calculate the numerical values of $\mathcal{F}_1$ and $\mathcal{G}_1$, the one-loop values of the coefficients defined in (3.1), we use the results found in [51, 22, 52] to evaluate 1-loop triangle diagrams for the topological current corresponding to the Pontryagin equation. In components, the supersymmetric expression $\Delta^- \Gamma' = 0$ on flat space translates into (in Euclidean signature)

$$\begin{aligned} \frac{1}{i}\frac{\delta}{\delta\tilde{\theta}}\Gamma' &= -\nabla^\mu J_\mu\,, \\ J_\mu &\equiv \frac{1}{8}\left(\varepsilon_{\mu\nu\rho\sigma}\left(A^{\nu a}\partial^\rho A^{\sigma a} + \frac{1}{3}A^{\sigma a}A^{\nu b}A^{\rho c}f_{abc}\right) - 2\bar{\lambda}^a\gamma_\mu\gamma_5\lambda^a\right). \end{aligned} \tag{A.4}$$

When coupled to the classical background fields $g_{\mu\nu}, V_\mu$, the above equation is anomalous such that

$$\frac{1}{i}\frac{\delta}{\delta\tilde{\theta}}\Gamma' = -\nabla_\mu J^\mu + \frac{g^2 k_1 N_V}{24\pi^2}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{g^2 k_2 N_V}{27\pi^2}V_{\mu\nu}\tilde{V}^{\mu\nu}\,. \tag{A.5}$$

This is the local component version of (3.1). First of all we note that although explicit factors of g^2 appear in the gravitational anomaly in (A.5), the anomaly coefficients k_1 and k_2 may be obtained to one-loop order by a simple triangle diagram computation. This is due to the unconventional normalization of the action (2.7), which implies that the R current and energy-momentum tensor involve a factor of $1/g^2$, whereas every propagator has a factor of g^2. Thus, for instance in the one-loop triangle $\langle JRR\rangle$ with J as in (A.4) there is a resulting overall factor of g^2. Moreover, a further important point is that the factor of 27 in the denominator of the second term is in agreement with [23]. When comparing with the anomaly equation for the R current derived in [22],

$$\nabla_\mu R^\mu = \frac{c-a}{24\pi^2}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{5a-3c}{9\pi^2}V_{\mu\nu}\tilde{V}^{\mu\nu}\,, \tag{A.6}$$

we see that the coefficients of the second anomaly term differ by a factor of 1/3. This takes account of the fact that there is no overall Bose symmetrization in $\langle JRR\rangle$ as opposed to $\langle RRR\rangle$.

Decomposing (3.1) into components gives the following relation between the coefficients in (3.1) and in (A.5):

$$\mathcal{F}_1 = \frac{1}{2}(5k_1 + k_2)\,, \qquad \mathcal{G}_1 = \frac{1}{2}(3k_1 + k_2)\,. \tag{A.7}$$

To determine k_1 and k_2, we calculate the three point correlation functions $\langle U_\mu T_{\nu\rho} T_{\lambda\sigma}\rangle$ and $\langle J_\mu R_\nu R_\rho\rangle$ to one-loop. To one loop order, the numerical values of k_1 and k_2 occurring in (A.5) have the following contributions:

$$k_1 = \frac{1}{8}\left(4 - 2\left(\frac{-1}{2}\right)\right)\left(\frac{1}{4}\right) = \frac{5}{32}\,. \tag{A.8}$$

The factors present have the following origin: As shown in [51], [9] and using the conventions of [22], the gauge field contribution to the divergence of $\langle JTT\rangle$ is $1/2 \equiv 4/8$, whereas the fermion contribution is $-1/8$. In the fermion contribution there is a factor of (-2) arising from the definition of J in (A.4) and a factor of 1/2 from the Majorana condition. Finally, there is an overall factor of 1/4 which is a remnant of the overall factor of 1/8 in (A.4). On the other hand, the one-loop value of k_2 is obtained using the results of [52] or [50] and

[9] For the abelian case the gauge contribution to the $R\tilde{R}$ term in the anomaly given by (A.6) has been calculated in [53].

is given by

$$k_2 = (-2)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{27}{8}\right) = -\left(\frac{27}{32}\right). \tag{A.9}$$

The (-2) followed by the (1/4) arises from the definition of the J_μ current (A.4), whereas the next (1/2) arises from the Majorana condition. The last factor is the anomaly coefficient arising in the divergence of $\langle JRR \rangle$.

These values for k_1, k_2, when inserted into (A.7), lead to the superspace coefficients given in (3.3).

A.3 Calculating the gravitational anomaly of the Konishi current to one loop

For calculating the gravitational anomalies contributing to the divergence of the Konishi current, which in components, again using the conventions of [22], reads

$$\nabla_\mu K^\mu = \frac{p_1 N_\chi}{24\pi^2} R_{\mu\nu\sigma\rho}\tilde{R}^{\mu\nu\sigma\rho} + \frac{p_2 N_\chi}{27\pi^2} V_{\mu\nu}\tilde{V}^{\mu\nu}\,, \tag{A.10}$$

with

$$K_\mu = \frac{1}{2}\bar{\psi}_i \gamma_\mu \gamma_5 \psi^i + \bar{\phi}_i \overleftrightarrow{D}_\mu \phi^i\,. \tag{A.11}$$

By decomposing the relevant supergravity expressions in (3.17) we have

$$\mathcal{H} = N_\chi \frac{1}{2}(5p_1 + p_2)\,, \qquad \mathcal{I} = N_\chi \frac{1}{2}(3p_1 + p_2)\,. \tag{A.12}$$

This is obtained by multiplying (3.17) with $(\bar{\mathcal{D}}^2 + R)$, subtracting its complex conjugate, decomposing into components, restricting to flat space and using the equations of motion.

Calculating the relevant one-loop triangle diagrams, we find, using the results of [50, 51] as before,

$$p_1 = -\frac{1}{16}\,, \qquad p_2 = \frac{3}{16}\,. \tag{A.13}$$

This gives

$$\mathcal{H}_1 = -\frac{1}{16}N_\chi\,, \qquad \mathcal{I}_1 = 0\,, \tag{A.14}$$

which is used in (3.18). The fact that $\mathcal{I} = 0$ at one loop explains why no term linear in the anomalous dimension γ contributes to the coefficient of the Euler density a.

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